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2007 J. Phys. A: Math. Theor. 40 7707

(http://iopscience.iop.org/1751-8121/40/27/019)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7707–7719 doi:10.1088/1751-8113/40/27/019

The (2+1)-dimensional nonisospectral relativistic Toda hierarchy related to the generalized discrete Painlevé hierarchy

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Received 23 October 2006, in final form 13 March 2007
Published 20 June 2007
Online at stacks.iop.org/JPhysA/40/7707

Abstract

In this paper, we will concentrate on the topic of integrable discrete hierarchies in 2+1 dimensions, and their connection with discrete Painlevé hierarchies. By considering a (2+1)-dimensional nonisospectral discrete linear problem, two new (2+1)-dimensional nonisospectral integrable lattice hierarchies—the 2+1 nonisospectral relativistic Toda lattice hierarchy and the 2+1 nonisospectral negative relativistic Toda lattice hierarchy—are constructed. It is shown that the reductions of the two new 2+1 nonisospectral lattice hierarchies lead to the (2+1)-dimensional nonisospectral Volterra lattice hierarchy and the (2+1)-dimensional nonisospectral negative Volterra lattice hierarchy. We also obtain two new (1+1)-dimensional nonisospectral integrable lattice hierarchies and two new ordinary difference hierarchies which are direct reductions of the two 2+1 nonisospectral integrable lattice hierarchies. One of the two difference hierarchies yields our previously obtained generalized discrete first Painlevé (dP_I) hierarchy and another one yields a generalized alternative discrete second Painlevé (alt-dP_{II}) hierarchy.

PACS numbers: 02.30.Hq, 02.30.Ik

1. Introduction

As is well known, there has been widespread interest in the study of continuous and discrete integrable systems because of their important roles in mathematics and physics. The well-known continuous integrable systems are the Korteweg–de Vries (KdV) equation and nonlinear Schrödinger equation, which have important physical applications. In the realm of discrete integrable systems described by nonlinear differential-difference or difference equations, perhaps the best known integrable discrete systems are the Toda lattice and the Volterra lattice. Remarkably, the Toda lattice itself has been a source of new integrable lattice, with new integrable lattices related to the Toda lattice having been proposed, e.g., the relativistic Toda lattice [1]. The relativistic Toda lattice was extensively studied. For example, its Lax pairs, the recursion operator, the Bäcklund transformation and the Hamiltonian structure have been obtained [2–6].

On the other hand, the investigation for multidimensional integrable systems is always an important and attractive topic. In continuous three-dimensional case, the Kadomtsev–Petviashvili (KP) equation [7], a natural two-dimensional generalization of the KdV, is well known. The KP arises in many field of physics, such as fluid mechanics, plasma physics, etc, and has all the usual properties of completely integrable systems. We also remark here that the self-dual Yang–Mills (SDYM) equations are of great importance in both physics and mathematics [8–10]. They play a central role in the field of integrable systems. It has been shown that three-dimensional reductions of the SDYM yield many equations including the KP, modified KP, (2+1)-dimensional N-wave, and Davey–Stewartson equationas [11–13]. Let us turn to multidimensional discrete integrable systems. In (2+1)-dimensional case, (2+1)-dimensional Toda lattice and its hierarchy and (2+1)-dimensional Volterra lattice are well known [14, 15]. Several other 2+1 lattices, for example, the so-called differential-difference KP equation due to Date, Jinbo and Miwa [16], 2+1 lattices constructed by Blaszak and Szum [17], were also studied.

We note that the most of known integrable systems (continuous or discrete, (1+1)-dimensional or multidimensional) relate to isospectral problems. Nonisospectral scattering problems, of course, since the work of Calogero [18], have continued to be the subject of much study, both in the continuous and discrete cases [19–24]. However, to the best of our knowledge, there is little work for multidimensional discrete nonisospectral flows. Very recently, we have proposed several (2+1)-dimensional integrable lattice hierarchies related to (2+1)-dimensional nonisospectral discrete linear problems [25–27]. One is a new (2+1)-dimensional nonisospectral Volterra lattice hierarchy and the second is a new (2+1)-dimensional nonisospectral extension of the discrete mKdV hierarchy. The third is a new (2+1)-dimensional nonisospectral Toda lattice hierarchy. We have also found that a generalized dP_{I} hierarchy and a generalized dP_{II} hierarchy can be obtained as stationary reductions of these new (2+1)-dimensional nonisospectral hierarchies. Thus we have greatly extended previously known results, for example that the dP_{I} and dP_{II} equations can be obtained from particular lattice equations [28], to the new result that generalized versions of the dP_{I} and dP_{II} hierarchies can be obtained from (2+1)-dimensional lattice hierarchies.

In the present paper, we will further concentrate on the topic of integrable discrete hierarchies in 2+1 dimensions, and their connection with discrete Painlevé hierarchies. By considering a (2+1)-dimensional nonisospectral discrete linear problem, two new (2+1)-dimensional nonisospectral integrable lattice hierarchies—2+1 nonisospectral relativistic Toda lattice hierarchy and 2+1 nonisospectral negative relativistic Toda lattice hierarchy—are constructed. It will be shown that the reductions of the two new 2+1 nonisospectral lattice hierarchies lead to our previously obtained (2+1)-dimensional nonisospectral Volterra lattice hierarchy and a new (2+1)-dimensional nonisospectral negative Volterra lattice hierarchy. We also obtain two new (1+1)-dimensional nonisospectral integrable lattice hierarchies and two new ordinary difference hierarchies which are direct reductions of the two 2+1 nonisospectral integrable lattice hierarchies. One of the two difference hierarchies yields our previously obtained generalized dP_{I} hierarchy and another one yields a generalized alt-dP_{II} hierarchy. We emphasize here that in the current paper we not only give two new (2+1)-dimensional integrable discrete nonisospectral flows, but also establish a connection between the 2+1 discrete nonisospectral flows and discrete Painlevé hierarchy. As we know, discrete Painlevé equations themselves have also physical applications. For example, the computation of a certain partition function in a model of two-dimensional quantum gravity led to dP_{I} [29, 30]. We thus think our results presented here will give new context in the area of multidimensional integrable discrete systems and discrete Painlevé hierarchy which is a subject undergone remarkable development in recent year.

2. (2+1)-dimensional nonisospectral relativistic Toda hierarchy

In this section, we construct a new (2+1)-dimensional nonisospectral relativistic Toda hierarchy by considering the following (2+1)-dimensional nonisospectral scattering problem:

$$E\psi_n(\lambda) = U_n(u_n, v_n, \lambda)\psi_n(\lambda), \tag{2.1}$$

$$\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}t} = \omega(\lambda)\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}y} + V_n^{(m)}(u_n, v_n, \lambda)\psi_n(\lambda), \tag{2.2}$$

where field functions u_n, v_n and wavefunction ψ_n are functions of arguments n, t, and y (with n being a discrete variable, and t and y continuous variables), and E is the shift operator, i.e., $Ef_n = f_{n+1}$, and where

$$U_n(u_n, v_n, \lambda) = \begin{pmatrix} \lambda + v_n & \lambda u_n \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

$$V_n^{(m)}(\lambda) = \begin{pmatrix} u_n A_n^{(m)}(\lambda) & -\lambda u_n B_n^{(m)}(\lambda) \\ E^{-1}B_n^{(m)}(\lambda) & u_{n-1}E^{-1}A_n^{(m)}(\lambda) + (\lambda + v_{n-1})E^{-1}B_n^{(m)}(\lambda) \end{pmatrix}. \tag{2.4}$$

Here time evolution of the spectral parameter $\lambda = \lambda(t, y)$ satisfies a nonisospectral condition

$$\lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda), \tag{2.5}$$

where $\omega(\lambda)$ and $\beta(\lambda)$ are two functions to be specified. To the best of our knowledge, the 2+1 discrete nonisospectral spectral problem, with derivation for a continuous variable y of wavefunction ψ_n appears in the temporal evolution equation, is new. In the continuous case, such type of 2+1 nonisospectral problems have been discussed. The first example is due to Calogero [18], and has as a subcase the equation

$$u_{xt} = u_{xxxy} + 4u_x u_{xy} + 2u_{xx}u_y, \tag{2.6}$$

which aries as the compatibility condition of the linear system

$$\psi_{xx} + (u_x - \lambda)\psi = 0, \qquad \psi_t = 4\lambda\psi_y + 2u_y\psi_x - u_{xy}\psi, \tag{2.7}$$

where the spectral parameter $\lambda = \lambda(y, t)$ satisfies the constraint [31, 32]

$$\lambda_t = 4\lambda\lambda_y. \tag{2.8}$$

Since the appearance of equation (2.6), many new continuous (2+1)-dimensional nonisospectral integrable hierarchies have been constructed (see e.g. references in [33]). Let us now discuss the spectral problem (2.1)–(2.2). The compatibility condition of the system (2.1)–(2.2) with (2.5) is

$$\frac{\partial U_n}{\partial t} + \beta(\lambda)\frac{\partial U_n}{\partial \lambda} - \omega(\lambda)\frac{\partial U_n}{\partial y} = V_{n+1}^{(m)}U_n - U_n V_n^{(m)}. \tag{2.9}$$

Our aim is to seek a proper matrix $V_n^{(m)}$ such that this nonisospectral discrete zero curvature equation yields a (2+1)-dimensional integrable lattice hierarchy. A direct calculation gives

$$v_{n,t} - \omega(\lambda)v_{n,y} + \beta(\lambda) = (\lambda + v_n)(E - 1)u_n A_n^{(m)} + \lambda(u_{n+1}E - u_n E^{-1})B_n^{(m)} \tag{2.10}$$

$$\beta(\lambda)u_n + \lambda(u_{n,t} - \omega(\lambda)u_{n,y}) = \lambda u_n\big[(u_{n+1}E - u_{n-1}E^{-1})A_n^{(m)} \\ + (E-1)\big((v_{n-1}+\lambda)E^{-1}B_n^{(m)}\big)\big]. \tag{2.11}$$

Then we expand $A_n^{(m)}$, $B_n^{(m)}$ as the following:

$$A_n^{(m)}(\lambda) = \sum_{j=1}^{m} a_j(n,t,y)\lambda^{m-j}, \qquad B_n^{(m)}(\lambda) = \sum_{j=1}^{m} b_j(n,t,y)\lambda^{m-j}, \tag{2.12}$$

and suppose $\omega(\lambda)$, $\beta(\lambda)$ have the forms

$$\omega(\lambda) = \lambda^m, \qquad \beta(\lambda) = \sum_{j=0}^{m-1} \beta_j \lambda^{m-j}, \qquad m \geqslant 1, \tag{2.13}$$

with $\beta_j (j = 0, 1, \ldots, m-1)$ being arbitrary constants. Substituting (2.12) and (2.13) to (2.10) and (2.11), and separating different powers of λ, we obtain the following equations:

$$u_{n,y} + u_n(1-E^{-1})b_1 = 0, \qquad (E-1)u_n a_1 = (u_n - u_{n+1}E^2)E^{-1}b_1 - v_{n,y} + \beta_0,$$
$$b_{j+1} = \alpha_{j+1}(t,y) - (E+1)u_n a_j - v_n b_j + (n+1)\beta_{j-1}, \qquad j = 1, 2, \ldots, m-1$$
$$(E-1)u_n a_{j+1} = \beta_j - v_n(E-1)u_n a_j + (u_n - u_{n+1}E^2)E^{-1}b_{j+1}, \qquad j = 1, 2, \ldots, m-1. \tag{2.14}$$

We can determine $a_j, b_j\, (j = 1, 2, \ldots, m)$ from above equations. Thus a new (2+1)-dimensional nonisospectral integrable lattice hierarchy is proposed,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = K \begin{pmatrix} a_m \\ b_m \end{pmatrix} - \begin{pmatrix} \beta_{m-1}u_n \\ 0 \end{pmatrix}, \qquad m \geqslant 1 \tag{2.15}$$

where K is a skew-symmetric matrix operator given by

$$K = \begin{pmatrix} u_n(E - E^{-1})u_n & u_n(1-E^{-1})v_n \\ v_n(E-1)u_n & 0 \end{pmatrix}. \tag{2.16}$$

We solve (2.14) as

$$a_1 = -\alpha_0(t,y) + u_n{}^{-1}(E-1)^{-1}(u_{n+1}E^2 - u_n)(E-1)^{-1}\frac{u_{n,y}}{u_n} \\ - u_n{}^{-1}(E-1)^{-1}v_{n,y} + \beta_0 n u_n{}^{-1}, \tag{2.17}$$
$$b_1 = \alpha_0(t,y) - (E-1)^{-1}\frac{u_{n+1,y}}{u_{n+1}},$$

and

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = J \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix} + G_k, \qquad k = 2, 3, \ldots, m, \tag{2.18}$$

where

$$G_k = \begin{pmatrix} \frac{n\beta_{k-1}}{u_n} - \frac{\beta_{k-2}}{u_n}(E-1)^{-1}u_{n+1} - n\beta_{k-2} - \alpha_{k-1} \\ (n+1)\beta_{k-2} + \alpha_{k-1} \end{pmatrix}, \tag{2.19}$$

$$J = \begin{pmatrix} J_{11} & J_{12} \\ -(E+1)u_n & -v_n \end{pmatrix} \tag{2.20}$$

with

$$J_{11} = u_n{}^{-1}(E-1)^{-1}[v_n(1-E) + (u_{n+1}E - u_nE^{-1})(E+1)]u_n$$
$$J_{12} = u_n{}^{-1}(E-1)^{-1}(u_{n+1}E - u_nE^{-1})v_n.$$

Thus we see that a_j, b_j can be solved for recursively, and lattice hierarchy (2.15) can be rewritten as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} + \begin{pmatrix} \beta_{m-1} u_n \\ 0 \end{pmatrix} = K J^{m-1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + K \left(\sum_{l=0}^{m-2} J^l G_{m-l} \right), \qquad m \geqslant 1. \tag{2.21}$$

Further we set another skew-symmetric matrix operator L given by

$$L = \begin{pmatrix} u_n{}^{-1}(E-1)^{-1}(u_{n+1}E^2 - u_n)(E-1)^{-1}u_n^{-1} & u_n{}^{-1}(1-E)^{-1} \\ (E^{-1}-1)^{-1}u_n^{-1} & 0 \end{pmatrix} \tag{2.22}$$

and

$$Q = KL = \begin{pmatrix} R - u_n(E-1)[v_{n-1}(E-1)^{-1}u_n^{-1}] & -u_n(1+E^{-1}) \\ v_n(u_{n+1}E^2 - u_n)(E-1)^{-1}u_n^{-1} & -v_n \end{pmatrix}, \tag{2.23}$$

where

$$R = u_n(1+E^{-1})(u_{n+1}E^2 - u_n)(E-1)^{-1}u_n^{-1} \tag{2.24}$$

is the recursion operator of the Volterra lattice hierarchy. Note that $L^{-1}JL = Q$, where

$$L^{-1} = \begin{pmatrix} 0 & u_n(E^{-1}-1) \\ (1-E)u_n & u_nE^{-1} - u_{n+1}E \end{pmatrix}. \tag{2.25}$$

Finally, we obtain the following (2+1)-dimensional integrable lattice hierarchy:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = Q^m \begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} + \sum_{l=0}^{m-1} \alpha_{m-l-1} Q^l K_1 + \sum_{l=-1}^{m-2} \beta_{m-l-2} Q^{l+1} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \qquad m \geqslant 1, \tag{2.26}$$

where

$$K_1 = \begin{pmatrix} u_n(u_{n-1} - u_{n+1} + v_n - v_{n-1}) \\ v_n(u_n - u_{n+1}) \end{pmatrix}.$$

This lattice hierarchy is a new (2+1)-dimensional nonisospectral relativistic Toda lattice hierarchy. The first term of the right-hand side of the equation corresponds to an extension of the relativistic Toda hierarchy to 2+1 dimensions; the second terms consists of a sum of standard (isospectral) relativistic Toda lattice flows. The third term consists of additional 1+1-dimensional nonisospectral terms. It is worth remarking here that the structure of the (2+1)-dimensional nonisospectral relativistic Toda lattice hierarchy is new and interesting. We now give the first and second flows of lattice hierarchy (2.26). Setting $m = 1$ and $m = 2$, we have, respectively,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_1} = Q \begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} + \alpha_0 K_1 + \beta_0 \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{2.27}$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_2} = Q^2 \begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} + (\alpha_1 + \alpha_0 Q)K_1 + (\beta_1 + \beta_0 Q) \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{2.28}$$

It is interesting to note that, under the reduction $v_n = 0$, hierarchy (2.26) reduces to our previously obtained (2+1)-dimensional nonisospectral Volterra lattice hierarchy:

$$u_{n,t_m} = R^m(u_{n,y}) + \sum_{l=0}^{m-1} \alpha_{m-l-1} R^l K_0 + \sum_{l=-1}^{m-2} \beta_{m-l-2} R^{l+1} u_n, \tag{2.29}$$

where the operator R is given by equation (2.24) and

$$K_0 = u_n(u_{n-1} - u_{n+1})$$

is a standard Volterra flow. We remark here that the following (1+1)-dimensional nonisospectral Volterra lattice hierarchy obtained in [22]

$$u_{n,t_m} = R^{m-1}(K_0 + u_n), \qquad m \geqslant 1, \tag{2.30}$$

is a special case of the new (2+1)-dimensional nonisospectral Volterra lattice hierarchy with $\partial_y = 0, \alpha_0 = \beta_0, \alpha_j = \beta_j = 0\,(j = 1, 2, \ldots, m-1)$. Our 2+1 lattice hierarchy (2.26) can also yield new 1+1 nonisospectral flows. Note that K^{-1} is given by

$$K^{-1} = \begin{pmatrix} 0 & u_n^{-1}(E-1)^{-1}v_n^{-1} \\ g_n & h_n \end{pmatrix}, \tag{2.31}$$

where

$$g_n = v_n^{-1}E(E-1)^{-1}u_n^{-1}, \qquad h_n = v_n^{-1}(E+1)(1-E)^{-1}v_n^{-1}, \tag{2.32}$$

and

$$Q^{-1} = L^{-1}K^{-1} = \begin{pmatrix} u_n(E^{-1}-1)g_n & u_n(E^{-1}-1)h_n \\ (u_nE^{-1} - u_{n+1}E)g_n & (u_nE^{-1} - u_{n+1}E)h_n - v_n^{-1} \end{pmatrix}. \tag{2.33}$$

Thus 2+1 relativistic Toda lattice hierarchy (2.26), under the reduction $\partial_{t_m} = 0$, leads to a new 1+1 nonisospectral lattice hierarchy

$$\begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} = -\sum_{l=0}^{m-1} \alpha_{m-l-1} Q^{l-m} K_1 - \sum_{l=-1}^{m-2} \beta_{m-l-2} Q^{l-m+1} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{2.34}$$

with corresponding nonisospectral linear problem

$$E\psi_n(\lambda) = U_n(u_n, v_n, \lambda)\psi_n(\lambda), \tag{2.35}$$

$$\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}y} = -\lambda^{-m} V_n^{(m)}(u_n, v_n, \lambda)\psi_n(\lambda). \tag{2.36}$$

Setting $v_n = 0$, lattice hierarchy (2.34) reduces to the nonisospectral flow:

$$u_{n,y} = -\sum_{l=0}^{m-1} \alpha_{m-l-1} R^{l-m} K_0 - \sum_{l=-1}^{m-2} \beta_{m-l-2} R^{l-m+1} u_n, \tag{2.37}$$

where the operator R^{-1} is defined by

$$R^{-1} = u_n(E-1)(u_{n+1}E^2 - u_n)^{-1}(E+1)^{-1}Eu_n^{-1}. \tag{2.38}$$

As we presented in the introduction, our purpose is not only constructing new multidimensional integrable discrete nonisospectral flows, but also exploring the connection between them and discrete Painlevé hierarchy. Next we will show that (2+1)-dimensional nonisospectral relativistic Toda hierarchy (2.26) encompasses a generalized $\mathrm{dP_I}$ hierarchy and an asymmetric $\mathrm{dP_I}$-like difference system. First we note that, under the reduction $\partial_{t_m} = \partial_y = 0$, hierarchy (2.26) reduces to the following new difference hierarchy:

$$\sum_{l=0}^{m-1} \alpha_{m-l-1} Q^{l} K_1 + \sum_{l=-1}^{m-2} \beta_{m-l-2} Q^{l+1} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad m \geqslant 1, \tag{2.39}$$

which relates to the linear problem

$$E\psi_n(\lambda) = U_n(u_n, v_n, \lambda)\psi_n(\lambda), \tag{2.40}$$

$$\left(\sum_{j=0}^{m-1}\beta_j\lambda^{m-j}\right)\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}\lambda}=V_n^{(m)}(u_n,v_n,\lambda)\psi_n(\lambda), \tag{2.41}$$

where u_n, v_n are only functions of n. Setting $v_n=0$, the new difference hierarchy yields a our previously obtained generalized $\mathrm{dP_I}$ hierarchy

$$\sum_{l=0}^{m-1}\alpha_{m-l-1}R^lK_0+\sum_{l=-1}^{m-2}\beta_{m-l-2}R^{l+1}u_n=0. \tag{2.42}$$

For $v_n\neq 0$, we consider the local case where $\beta_{m-l-2}=0, l=0,1,\dots,m-2,$

$$\sum_{l=0}^{m-1}\alpha_{m-l-1}Q^lK_1+\beta_{m-1}\begin{pmatrix}u_n\\v_n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, m\geqslant 1. \tag{2.43}$$

This equation can also be written as

$$K\begin{pmatrix}F_{m,n}\\G_{m,n}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\qquad \begin{pmatrix}F_{m,n}\\G_{m,n}\end{pmatrix}=\sum_{j=0}^{m-1}\alpha_{m-j-1}L_{j+1,n}+\beta_{m-1}\begin{pmatrix}\frac{n}{u_n}\\\frac{-n}{v_n}\end{pmatrix}, \tag{2.44}$$

where K is the Hamiltonian operator (2.16) of the relativistic Toda lattice hierarchy, and $L_{j+1,n}$ satisfies the following equation:

$$Q^jK_1=K_{j+1}=KL_{j+1,n},\qquad j\geqslant 0. \tag{2.45}$$

Note that $Q=KL$, we have the recursion relation

$$KL_{j,n}=L^{-1}L_{j+1,n},\qquad j\geqslant 1, \tag{2.46}$$

where L^{-1} is another Hamiltonian operator (2.25) of the relativistic Toda lattice hierarchy. We solve the recursion relation as

$$L_{1,n}=\begin{pmatrix}-1\\1\end{pmatrix},\qquad L_{2,n}=\begin{pmatrix}v_{n-1}+v_n-u_{n-1}-u_n-u_{n+1}\\u_n+u_{n+1}-v_n\end{pmatrix}, \tag{2.47}$$

and

$$L_{3,n}=\begin{pmatrix}au_{n-1}+2u_n(v_{n-1}+v_n)+bu_{n+1}-\left(v_{n-1}^2+v_n^2+v_{n-1}v_n\right)-\left(u_{n-1}^2+u_n^2+u_{n+1}^2\right)\\u_n(u_{n+1}+u_n+u_{n-1}-2v_n-v_{n-1})+u_{n+1}(u_{n+2}+u_{n+1}+u_n-2v_n-v_{n+1})+v_n^2\end{pmatrix}, \tag{2.48}$$

where

$$a=v_{n-2}+2v_{n-1}+v_n-u_{n-2}-2u_n,\qquad b=v_{n-1}+2v_n+v_{n+1}-u_{n-1}-2u_n-u_{n+2}.$$

Note that

$$(E-1)\begin{pmatrix}I_{m,n}\\J_{m,n}\end{pmatrix}=\begin{pmatrix}0&v_n^{-1}\\u_n^{-1}&0\end{pmatrix}K\begin{pmatrix}F_{m,n}\\G_{m,n}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{2.49}$$

where

$$\begin{pmatrix}I_{m,n}\\J_{m,n}\end{pmatrix}=\begin{pmatrix}u_nF_{m,n}\\(1+E^{-1})(u_nF_{m,n})+E^{-1}(v_nG_{m,n}),\end{pmatrix}, \tag{2.50}$$

we thus see that our difference hierarchy (2.43), or equivalently (2.44) can always be summed to give

$$u_nF_{m,n}+c_m=0, \tag{2.51}$$

$$(1+E^{-1})(u_nF_{m,n})+E^{-1}(v_nG_{m,n})+d_m=0, \tag{2.52}$$

where c_m and d_m are two arbitrary constants, or equivalently

$$u_n F_{m,n} + c_m = 0, \tag{2.53}$$

$$E^{-1}(v_n G_{m,n}) + d_m - 2c_m = 0. \tag{2.54}$$

Let us give an example of the case $m = 2$. We have

$$\begin{pmatrix} F_{2,n} \\ G_{2,n} \end{pmatrix} = \alpha_0 L_{2,n} + \alpha_1 L_{1,n} + \beta_1 \begin{pmatrix} \frac{n}{u_n} \\ \frac{-n}{v_n} \end{pmatrix}, \tag{2.55}$$

and thus obtain the second-order system

$$u_{n+1} + u_n = \frac{\beta_1 n + 2c_2 - d_2}{\alpha_0 v_n} + v_n - \frac{\alpha_1}{\alpha_0} \tag{2.56}$$

$$v_n + v_{n-1} = \frac{-(\beta_1 n + c_2)}{\alpha_0 u_n} + u_{n+1} + u_n + u_{n-1} + \frac{\alpha_1}{\alpha_0}. \tag{2.57}$$

This last system is an asymmetric $\mathrm{dP_I}$-like difference equation.

3. (2+1)-dimensional nonisospectral negative relativistic Toda hierarchy

As we know, negative relativistic Toda lattice hierarchy can be constructed along with the same linear spectral problem as relativistic Toda lattice hierarchy. This motivates us to construct a (2+1)-dimensional nonisospectral negative relativistic Toda hierarchy along with the nonisospectral scattering problem (2.1)–(2.2). We suppose that time evolution of spectral parameter $\lambda(t, y)$ satisfies the following nonisospectral condition:

$$\lambda_t = \lambda^{-m}\lambda_y + \sum_{j=1}^{m} \gamma_j \lambda^{j-m}, \qquad m \geqslant 1, \tag{3.1}$$

and we set, in matrix $V_n^m(\lambda)$, that

$$A_n^{(m)}(\lambda^-) = \sum_{j=1}^{m} \bar{a}_{m-j}(n, t, y)\lambda^{-j}, \qquad B_n^{(m)}(\lambda^-) = \sum_{j=1}^{m} \bar{b}_{m-j}(n, t, y)\lambda^{-j}. \tag{3.2}$$

From the nonisospectral discrete zero curvature equation, we obtain the following equations:

$$v_n(E-1)(u_n\bar{a}_0) + v_{n,y} = 0, \qquad (E - E^{-1})u_n\bar{a}_0 + (E-1)v_{n-1}E^{-1}\bar{b}_0 + \frac{u_{n,y}}{u_n} - \gamma_1 = 0,$$

$$v_n(E-1)(u_n\bar{a}_{j+1}) + (E-1)(u_n\bar{a}_j) + (u_{n+1}E - u_nE^{-1})\bar{b}_j = \gamma_{j+1}, \qquad j = 0, 1, \ldots, m-2$$

$$v_n\bar{b}_{j+1} = n\gamma_{j+2} + \mu_j - (1+E)u_n\bar{a}_{j+1} - \bar{b}_j, \qquad j = 0, 1, \ldots, m-2. \tag{3.3}$$

Here $\bar{a}_j$ and $\bar{b}_j$ $(j = 0, 1, \ldots, m-1)$ are to be determined. This then yields the following (2+1)-dimensional nonisospectral lattice hierarchy:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = \bar{K}\begin{pmatrix} \bar{a}_{m-1} \\ \bar{b}_{m-1} \end{pmatrix} - \begin{pmatrix} 0 \\ \gamma_m \end{pmatrix}, \qquad m \geqslant 1, \tag{3.4}$$

where $\bar{K}$ is a skew-symmetric matrix operator given by

$$\bar{K} = \begin{pmatrix} 0 & u_n(1 - E^{-1}) \\ (E-1)u_n & u_{n+1}E - u_nE^{-1} \end{pmatrix}. \tag{3.5}$$

We see that $\bar{a}_j$ and $\bar{b}_j$ can be solved for recursively as

$$\begin{aligned}\bar{a}_0(n,t,y) &= \frac{\gamma}{u_n} - \frac{1}{u_n}(E-1)^{-1}\frac{v_{n,y}}{v_n},\\ \bar{b}_0(n,t,y) &= \frac{\alpha_0+n\gamma_1}{v_n} + \frac{1}{v_n}(E-1)^{-1}\left[\frac{v_{n,y}}{v_n}+\frac{v_{n+1,y}}{v_{n+1}}-\frac{u_{n+1,y}}{u_{n+1}}\right],\end{aligned} \tag{3.6}$$

and

$$\begin{pmatrix}\bar{a}_{k-1}\\ \bar{b}_{k-1}\end{pmatrix} = \bar{J}\begin{pmatrix}\bar{a}_{k-2}\\ \bar{b}_{k-2}\end{pmatrix} + \bar{G}_{k-2}, \qquad k \geqslant 2, \tag{3.7}$$

where

$$\bar{J} = \begin{pmatrix} \frac{-1}{u_n}(E-1)^{-1}\big(\frac{1}{v_n}(E-1)\big)u_n & \frac{-1}{u_n}(E-1)^{-1}\frac{1}{v_n}(u_{n+1}E-u_nE^{-1}) \\ \frac{1}{v_n}(E+1)(E-1)^{-1}\big(\frac{1}{v_n}(E-1)\big)u_n & \frac{-1}{v_n}\big[1-(E+1)(E-1)^{-1}\frac{1}{v_n}(u_{n+1}E-u_nE^{-1})\big]\end{pmatrix}, \tag{3.8}$$

and

$$\bar{G}_{k-2} = \begin{pmatrix} \frac{\gamma_{k-1}}{u_n}(E-1)^{-1}v_n^{-1} \\ \frac{1}{v_n}\big[n\gamma_k+\mu_{k-2}-\gamma_{k-1}(E+1)(E-1)^{-1}v_n^{-1}\big]\end{pmatrix}. \tag{3.9}$$

Thus (2+1)-dimensional lattice hierarchy (3.4) can be rewritten in the form

$$\begin{pmatrix}u_n\\ v_n\end{pmatrix}_{t_m} = \bar{K}\bar{J}^{m-1}\begin{pmatrix}\bar{a}_0\\ \bar{b}_0\end{pmatrix} - \begin{pmatrix}0\\ \gamma_m\end{pmatrix} + \bar{K}\left(\sum_{l=0}^{m-2}\bar{J}^l\bar{G}_{m-l-2}\right), \qquad m \geqslant 1. \tag{3.10}$$

Further, setting another skew-symmetric matrix operator $\bar{L}$ given by

$$\bar{L} = \begin{pmatrix} 0 & u_n^{-1}(1-E)^{-1}v_n^{-1} \\ v_n^{-1}(E^{-1}-1)^{-1}u_n^{-1} & v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\end{pmatrix}, \tag{3.11}$$

and

$$\bar{Q} = \bar{K}\bar{L} = \begin{pmatrix} u_n(E^{-1}-1)g_n & u_n(1-E^{-1})h_n \\ (u_nE^{-1}-u_{n+1}E)g_n & (u_{n+1}E-u_nE^{-1})h_n - v_n^{-1}\end{pmatrix} \tag{3.12}$$

and noting that $\bar{L}^{-1}\bar{J}\bar{L} = \bar{Q}$, where

$$\bar{L}^{-1} = \begin{pmatrix} u_n(E+1)(E^{-1}-1)u_n & u_n(E^{-1}-1)v_n \\ v_n(1-E)u_n & 0\end{pmatrix}, \tag{3.13}$$

we finally obtain the following (2+1)-dimensional lattice hierarchy:

$$\begin{aligned}\begin{pmatrix}u_n\\ v_n\end{pmatrix}_{t_m} &= \bar{Q}^m\begin{pmatrix}u_{n,y}\\ v_{n,y}\end{pmatrix} + \sum_{l=0}^{m-1}\mu_{m-l-2}\bar{Q}^l\begin{pmatrix}u_n(1-E^{-1})v_n^{-1}\\ (u_{n+1}E-u_nE^{-1})v_n^{-1}\end{pmatrix}\\ &\quad + \sum_{l=0}^{m-1}\gamma_{m-l}\bar{Q}^l\begin{pmatrix}u_n(1-E^{-1})(nv_n^{-1})\\ (u_{n+1}E-u_nE^{-1})(nv_n^{-1})\end{pmatrix} - \begin{pmatrix}0\\ \gamma_m\end{pmatrix}\\ &\quad + \sum_{l=0}^{m-2}\gamma_{m-l-1}\bar{Q}^l\begin{pmatrix} u_n(E^{-1}-1)\big[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\big] \\ v_n^{-1}+(u_nE^{-1}-u_{n+1}E)\big[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\big]\end{pmatrix},\end{aligned} \tag{3.14}$$

where $\mu_{-1} = \alpha_0$. We find that the hierarchy is a (2+1)-dimensional nonisospectral extension of the negative relativistic Toda lattice hierarchy. The first term of the right-hand side of equation (3.14) corresponds to an extension of the negative relativistic Toda hierarchy to

2+1 dimensions; the second terms consists of the standard isospectral negative relativistic Toda lattice flows. The other terms consist of additional (1+1)-dimensional nonisospectral. Let us give the first flow of our hierarchy

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_1} = \bar{Q}\begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} + \alpha_0 \begin{pmatrix} u_n(1-E^{-1})v_n^{-1} \\ (u_{n+1}E - u_nE^{-1})v_n^{-1} \end{pmatrix} + \gamma_1 \begin{pmatrix} u_n(1-E^{-1})(nv_n^{-1}) \\ (u_{n+1}E - u_nE^{-1})(nv_n^{-1}) - 1 \end{pmatrix}. \tag{3.15}$$

Noting $\bar{Q}^{-1} = Q$, we see that lattice hierarchy (3.14), under the reduction $\partial_{t_m} = 0$, yields the following new (1+1)-dimensional nonisospectral lattice hierarchy:

$$\begin{aligned} \begin{pmatrix} u_{n,y} \\ v_{n,y} \end{pmatrix} = &-\sum_{l=0}^{m-1} \mu_{m-l-2} Q^{m-l} \begin{pmatrix} u_n(1-E^{-1})v_n^{-1} \\ (u_{n+1}E - u_nE^{-1})v_n^{-1} \end{pmatrix} \\ &- \sum_{l=0}^{m-1} \gamma_{m-l} Q^{m-l} \begin{pmatrix} u_n(1-E^{-1})(nv_n^{-1}) \\ (u_{n+1}E - u_nE^{-1})(nv_n^{-1}) \end{pmatrix} + Q^m \begin{pmatrix} 0 \\ \gamma_m \end{pmatrix} \\ &- \sum_{l=0}^{m-2} \gamma_{m-l-1} Q^{m-l} \begin{pmatrix} u_n(E^{-1}-1)\left[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\right] \\ v_n^{-1} + (u_nE^{-1} - u_{n+1}E)\left[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\right] \end{pmatrix}, \end{aligned} \tag{3.16}$$

which relates to the nonisospectral linear problem

$$E\psi_n(\lambda) = U_n(u_n, v_n, \lambda)\psi_n(\lambda), \tag{3.17}$$

$$\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}y} = -\lambda^m V_n^{(m)}(u_n, v_n, \lambda)\psi_n(\lambda). \tag{3.18}$$

Let us consider the special case of $v_n = 0$. Under this case, equation (3.3) reduces to

$$\begin{aligned} &(E-E^{-1})u_n\bar{a}_0 + \frac{u_{n,y}}{u_n} - \gamma_1 = 0, \\ &(E-1)(u_n\bar{a}_j) + (u_{n+1}E - u_nE^{-1})\bar{b}_j = \gamma_{j+1}, \qquad j = 0, 1, \ldots, m-2 \\ &(1+E)u_n\bar{a}_{j+1} + \bar{b}_j = n\gamma_{j+2} + \mu_j \qquad j = 0, 1, \ldots, m-2, \end{aligned} \tag{3.19}$$

and also the following equation is satisfied:

$$(E-1)(u_n\bar{a}_{m-1}) + (u_{n+1}E - u_nE^{-1})\bar{b}_{m-1} = \gamma_m. \tag{3.20}$$

Hierarchy (3.4) thus reduces to 2+1 lattice hierarchy

$$(u_n)_{t_m} = u_n(1-E^{-1})\bar{b}_{m-1}. \tag{3.21}$$

We find that the 2+1 lattice hierarchy is a (2+1)-dimensional nonisospectral negative Volterra hierarchy. Let us give explanation for the fact. We suppose that the field function u_n is expressed by the tau-function

$$u_n = \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}, \tag{3.22}$$

and then equation hierarchy (3.21) can be rewritten as

$$\left(\ln\frac{\tau_{n+1}}{\tau_{n-1}}\right)_{t_m} = \bar{b}_{m-1}, \qquad m \geqslant 1. \tag{3.23}$$

We solve equation (3.19) as

$$\bar{b}_0 = \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}}\left[\delta_0 + (-1)^n\mu_0 + \frac{\gamma_1}{2}(E^2-1)^{-1}\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n} - (E^2-1)^{-1}\left[\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n}\left((-1)^n\zeta_0 - (E+1)^{-1}\frac{u_{n+1,y}}{u_{n+1}}\right)\right]\right]$$

$$\bar{b}_{m-1} = \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}}\left[\delta_{m-1} + (-1)^n\mu_{m-1} + \frac{\gamma_m}{2}(E^2-1)^{-1}\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n} + (E^2-1)^{-1}\left(\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n}HE\bar{b}_{m-2}\right)\right], \qquad m \geqslant 2, \tag{3.24}$$

where the operator H is defined by $H = (E+1)^{-1}(E-1)$. Taking $\zeta_0 = \mu_0 = \gamma_1 = 0$, $\delta_0 = 1$, $m = 1$ and $\partial_y = 0$, we get the first member of hierarchy (3.23)

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}}, \tag{3.25}$$

which is the simplest flow of the negative Volterra hierarchy [34]. The second flow with $\zeta_0 = \mu_0 = \mu_1 = \gamma_1 = \gamma_2 = \delta_1 = 0$, $\delta_0 = 1$, and $\partial_y = 0$ of hierarchy (3.23) is the second one of the negative Volterra hierarchy

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}}(E^2-1)^{-1}\left[\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n}H\frac{\tau_{n+1}^2}{\tau_{n+2}\tau_n}\right]. \tag{3.26}$$

Therefore, hierarchy (3.21) is really a nonisospectral 2+1 extension of the negative Volterra hierarchy. Let us show how the new (2+1)-dimensional nonisospectral negative relativistic Toda hierarchy is connected to the discrete Painlevé hierarchy. It is obvious that 2+1 negative relativistic Toda lattice hierarchy (3.14), under the reduction $\partial_{t_m} = \partial_y = 0$, yields the following new difference hierarchy:

$$\begin{pmatrix}0\\ \gamma_m\end{pmatrix} = \sum_{l=0}^{m-1}\mu_{m-l-2}\bar{Q}^l\begin{pmatrix}u_n(1-E^{-1})v_n^{-1}\\ (u_{n+1}E - u_nE^{-1})v_n^{-1}\end{pmatrix} + \sum_{l=0}^{m-1}\gamma_{m-l}\bar{Q}^l\begin{pmatrix}u_n(1-E^{-1})(nv_n^{-1})\\ (u_{n+1}E - u_nE^{-1})(nv_n^{-1})\end{pmatrix} + \sum_{l=0}^{m-2}\gamma_{m-l-1}\bar{Q}^l\begin{pmatrix}u_n(E^{-1}-1)\left[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\right]\\ v_n^{-1} + (u_nE^{-1} - u_{n+1}E)\left[v_n^{-1}(E+1)(E-1)^{-1}v_n^{-1}\right]\end{pmatrix}, \tag{3.27}$$

with the corresponding linear problem

$$E\psi_n(\lambda) = U_n(u_n, v_n, \lambda)\psi_n(\lambda), \tag{3.28}$$

$$\left(\sum_{j=1}^{m}\gamma_j\lambda^{j-m}\right)\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}\lambda} = V_n^{(m)}(u_n, v_n, \lambda)\psi_n(\lambda), \tag{3.29}$$

where u_n, v_n are the only functions of n. We give the member of difference hierarchy (3.27) with $m = 2$ and $\gamma_1 = 0$:

$$\begin{pmatrix}0\\ \gamma_2\end{pmatrix} = \alpha_0\bar{Q}\begin{pmatrix}u_n(1-E^{-1})v_n^{-1}\\ (u_{n+1}E - u_nE^{-1})v_n^{-1}\end{pmatrix} - \begin{pmatrix}u_n(1-E^{-1})\left(\frac{n\gamma_2+\mu_0}{v_n}\right)\\ (u_{n+1}E - u_nE^{-1})\left(\frac{n\gamma_2+\mu_0}{v_n}\right)\end{pmatrix}. \tag{3.30}$$

Solving the difference system, we obtain

$$\alpha_0\left(\frac{u_n}{v_{n-1}v_n}+\frac{u_{n+1}}{v_nv_{n+1}}\right)=kv_n+\frac{\alpha_0}{v_n}-\gamma_2n-\mu_0 \tag{3.31}$$

$$u_n=\frac{(\gamma_2n+\alpha_1)v_{n-1}v_n}{kv_{n-1}v_n-\alpha_0}. \tag{3.32}$$

Taking $k=-\alpha_0$ and eliminating u_n from this system yields

$$\frac{z_{n-1}}{v_{n-1}v_n+1}+\frac{z_n}{v_nv_{n+1}+1}=\alpha_0\left(-v_n+\frac{1}{v_n}\right)+z_n+\mu, \tag{3.33}$$

where $z_{n-1}=-\gamma_2n-\alpha_1$. This last equation is the alt-dP_{II} equation. We thus see that (2+1)-dimensional nonisospectral negative relativistic Toda hierarchy encompasses a generalized alt-dP_{II} hierarchy.

4. Conclusions and discussions

We have given two new (2+1)-dimensional nonisospectral lattice hierarchies—2+1 nonisospectral relativistic Toda lattice hierarchy and 2+1 nonisospectral negative relativistic Toda lattice hierarchy. We have shown that a generalized dP_I hierarchy and a generalized alt-dP_{II} hierarchy can be embedded two new ordinary difference hierarchies which are reductions of the two 2+1 nonisospectral lattice flows. We have also presented that they yield new integrable hierarchies, including a 2+1 nonisospectral Volterra lattice hierarchy, a 2+1 nonisospectral negative Volterra lattice hierarchy and new (1+1)-dimensional nonisospectral hierarchies. The method of constructing the two new (2+1)-dimensional integrable nonisospectral lattice hierarchies is the nonisospectral scattering method. By using it, we have succeeded in constructing several new (2+1)-dimensional integrable nonisospectral lattice hierarchies. We believe that many more 2+1 integrable nonisospectral lattice hierarchies can be given in this way. It would be interesting to exploit other integrability for the two new 2+1 nonisospectral lattice hierarchies, such as infinitely many conservation laws and soliton solutions. We will leave the topic to the future.

Acknowledgments

The work is supported by the National Natural Science Foundation of China under grant no. 10671125, and the Shanghai Pujiang Program, and also in part by the Junta de Castilla y León under contract SA011/04. I warmly thank Dr. P R Gordoa and Dr. A Pickering for many illuminating discussions. I am also very grateful to the referees for their useful remarks on the original manuscript.

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